

Forward Backward Stochastic Differential Equations - Asymptotics and a Large Deviations Principle

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Abstract

We study the asymptotic behaviour of solutions of Forward Backward Stochastic Differential Equations in the coupled case, when the diffusion coefficient of the forward equation is multiplicatively perturbed by a small parameter that converges to zero. Furthermore, we establish a Large Deviation Principle for the laws of the corresponding processes.

1 Introduction

Forward-Backward Stochastic Differential Equations (FBSDEs for short) is a subject in Stochastic Analysis that is being increasingly studied by the mathematical community in the recent years.

Initially FBSDEs become known due to Stochastic Optimal Control Theory, with the pionering work [3], and, after this kind of stochastic equations were developed in various fields, such as Mathematical Finance or Mathematical Physics. The connections between Stochastic Differential Equations and Partial Differential Equations are well known since the celebrated Feynman-Kac formula. In this perspective, Peng [17] realized that a solution of a Backward Stochastic Differential Equation could be used as a probabilistic interpretation for the solutions of a huge class of quasilinear parabolic PDEs.

In this work, we present the asymptotic study and the establishment of a Large Deviations Principle for the laws of the solutions of the Forward Backward Stochastic Differential Equations in the fully coupled case. One of the main points is the independence of PDEs results in the asymptotic study

of the FBSDE. These results only use probabilistic arguments. They are local in time, although, it is also possible to establish global results in time, as we remark in Section 5. In this case the methods rely on PDE results. We present the asymptotic study of FBSDEs solutions when the diffusion coefficient of the forward equation approaches zero. This question addresses the problem of the convergence of classical/viscosity solutions of the quasilinear parabolic system of PDEs associated to the Backward Equation. When this quasilinear parabolic system of PDEs takes the form of the Backward Burgers Equation, the problem becomes the convergence of the solution when the viscosity parameter converges to zero. This study is the object of Section 3.

In Section 4, using the classical tools of the Freidlin-Wentzell Theory and a Simple Version of the "Contraction Principle" (cf [7]), a Large Deviations Principle for the laws of the corresponding Forward Backward Processes of the system is established.

2 Preliminaries

Let $T > 0$, $d, k \in \mathbb{N}$ and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$ be a complete filtered probability space. On such a space we consider a d - dimensional Brownian Motion $(B_t)_{0 \leq t \leq T}$ such that $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ is the natural filtration of $(B_t)_{0 \leq t \leq T}$, augmented with the collection of the null sets of Ω , $\mathcal{N} := \{A \subset \Omega : \exists G \in \mathcal{F} \text{ such that } \mathbb{P}(G) = 0\}$.

We define the following functional spaces, given $t \in [0, T]$:

$S_t^2(\mathbb{R}^d)$, the space of the continuous $\{\mathcal{F}_t\}_{t \leq s \leq T}$ adapted processes $\phi : \Omega \times [t, T] \longrightarrow \mathbb{R}^d$ such that

$$\|\phi\|_\infty^2 = \mathbb{E} \sup_{t \leq s \leq T} |\phi_s|^2 < \infty$$

and $H_t^2(\mathbb{R}^{k \times d})$, the space of the $\{\mathcal{F}_t\}_{t \leq s \leq T}$ adapted measurable processes $\phi : \Omega \times [t, T] \longrightarrow \mathbb{R}^{k \times d}$ such that

$$\|\phi\|_2^2 = \mathbb{E} \int_t^T |\phi_s|^2 ds.$$

These spaces are naturally complete normed spaces.

The goal of this work is to study the asymptotic behaviour when $\varepsilon \rightarrow 0$ of the Forward Backward Stochastic Differential Equations (FBSDE for short)

$$\begin{cases} X_s^{t,\varepsilon,x} = x + \int_t^s f(r, X_r^{t,\varepsilon,x}, Y_r^{t,\varepsilon,x})dr + \sqrt{\varepsilon} \int_t^s \sigma(r, X_r^{t,\varepsilon,x}, Y_r^{t,\varepsilon,x})dB_r \\ Y_s^{t,\varepsilon,x} = h(X_T^{t,\varepsilon,x}) + \int_s^T g(r, X_r^{t,\varepsilon,x}, Y_r^{t,\varepsilon,x}, Z_r^{t,\varepsilon,x})dr - \int_s^T Z_r^{t,\varepsilon,x}dB_r \\ x \in \mathbb{R}^d; \quad 0 \leq t \leq s \leq T \end{cases} \quad (2.1)$$

where

$$\begin{aligned} f &: [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \rightarrow \mathbb{R}^d \\ g &: [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^k \\ \sigma &: [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \rightarrow \mathbb{R}^{d \times d} \\ h &: \mathbb{R}^k \rightarrow \mathbb{R}^k \end{aligned}$$

are measurable functions with respect to the Borelian fields in the Euclidean spaces where they are defined. If we assume the classical regularity assumptions on the coefficients of the FBSDE (that we present in section 2), theorem 1.1 of [5] ensures that (2.1) has a unique local solution in $\mathcal{M}_t = S_t^2(\mathbb{R}^d) \times S_t^2(\mathbb{R}^k) \times H_t^2(\mathbb{R}^{k \times d})$. The space \mathcal{M}_t is a Banach Space for the natural norm associated to the product topology structure.

The solution will be denoted $(X_s^{t,\varepsilon,x}, Y_s^{t,\varepsilon,x}, Z_s^{t,\varepsilon,x})_{t \leq s \leq T}$, or sometimes only $(X_s^\varepsilon, Y_s^\varepsilon, Z_s^\varepsilon)_{t \leq s \leq T}$ for simplicity.

In [15] it is shown, with probabilistic techniques, that $u^\varepsilon(t, x) = Y_t^{t,\varepsilon,x}$, which is a deterministic vector by *Blumenthal's 0-1 Law* (remark 1.2 of [5]), is a viscosity solution of the associated quasilinear parabolic partial differential equation

$$\begin{cases} \frac{\partial(u^\varepsilon)^l}{\partial t}(t, x) + \frac{\varepsilon}{2} \sum_{i=1}^d a_{ij}(t, x, u^\varepsilon(t, x)) \frac{\partial^2(u^\varepsilon)^l}{\partial x_i \partial x_j}(t, x) + \\ \sum_{i=1}^d f_i^l(t, x, u^\varepsilon(t, x)) \frac{\partial(u^\varepsilon)^l}{\partial x_i}(t, x) + \\ g^l(t, x, u^\varepsilon(t, x), \sqrt{\varepsilon} \nabla_x u^\varepsilon(t, x) \sigma^\varepsilon(t, x, u^\varepsilon(t, x))) = 0 \\ u^\varepsilon(T, x) = h(x) \\ x \in \mathbb{R}^d; \quad t \in [0, T]; \quad l = 1, \dots, k \end{cases} \quad (2.2)$$

where $a_{i,j} = (\sigma \sigma^T)_{i,j}$.

We define below the notion of viscosity solution for the parabolic system of Partial Differential Equations (PDEs for short) (2.2).

For each $\varepsilon > 0$, consider the following differential operator,

$$(L_l^\varepsilon \varphi)(t, x, y, z) = \frac{\varepsilon}{2} \sum_{i,j=1}^d a_{ij}(t, x, y) \frac{\partial^2 \varphi^l}{\partial x_i \partial x_j}(t, x) + \langle f(t, x, y), \nabla \varphi^l(t, x) \rangle$$

$$l = 1, \dots, k \quad \forall \varphi \in C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R}^k), \quad t \in [0, T], \quad (x, y, z) \in \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$$

The space $C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R}^k)$ is the space of the functions $\phi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^k$, which are C^1 with respect to the first variable and C^2 with respect to the second variable.

The system (2.2) reads

$$\begin{cases} \frac{\partial (u^\varepsilon)^l}{\partial t} + (L_l^\varepsilon u^\varepsilon)(t, x, u^\varepsilon(t, x), \nabla_x u^\varepsilon(t, x) \sigma(t, x, u^\varepsilon(t, x))) + \\ g^l(t, x, u^\varepsilon(t, x), \sqrt{\varepsilon} \nabla_x u^\varepsilon(t, x) \sigma(t, x, u^\varepsilon(t, x))) = 0 \\ u^\varepsilon(T, x) = h(x), \quad x \in \mathbb{R}^d, \quad t \in [0, T], \quad l = 1, \dots, k \end{cases} \quad (2.3)$$

Definition 2.1. Viscosity solution

Let $u^\varepsilon \in C([0, T] \times \mathbb{R}^d, \mathbb{R}^k)$. The function u^ε is said to be a viscosity sub-solution (resp. super-solution) of the system (2.2) if

$$(u^\varepsilon)^l(T, x) \leq h^l(x); \quad \forall l = 1, \dots, k; \quad x \in \mathbb{R}^d$$

$$(\text{resp. } (u^\varepsilon)^l(T, x) \geq h^l(x); \quad \forall l = 1, \dots, k; \quad x \in \mathbb{R}^d)$$

and for each $l = 1, \dots, k$, $(t, x) \in [0, T] \times \mathbb{R}^d$, $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R}^k)$ such that (t, x) is a local minimum (resp. maximum) point of $\varphi - u^\varepsilon$, we have

$$\begin{aligned} & \frac{\partial \varphi}{\partial t}(t, x) + (L_l^\varepsilon \varphi)(t, x, u^\varepsilon(t, x), \sqrt{\varepsilon} \nabla_x \varphi(t, x) \sigma(t, x, u^\varepsilon(t, x))) + \\ & g^l(t, x, u^\varepsilon(t, x), \sqrt{\varepsilon} \nabla_x \varphi(t, x) \sigma(t, x, u^\varepsilon(t, x))) \geq 0; \quad l = 1, \dots, k \\ & (\text{resp. } \leq 0). \end{aligned}$$

The function u^ε is said to be a viscosity solution of the system (2.2) if u^ε is both a viscosity sub-solution and a viscosity super-solution of this system.

Under more restrictive assumptions (that we present in Section 3), we shall have

$$u^\varepsilon(t, x) = Y_t^{t, \varepsilon, x} \quad (2.4)$$

and u^ε will be actually a classical solution of (2.2). This can be proved using the *Four Step Scheme Methodology* of *Ma-Protter-Young* [14] with the help of *Ladyzhenskaja's* work in quasilinear parabolic PDEs [12]. If $f(s, x, y) = y$, $k = d$, and $a_{i,j} = (\sigma\sigma^T)_{i,j} = I_{d \times d}$, the system (2.2) becomes a Backward Burgers Equation

$$\begin{cases} \frac{\partial u^\varepsilon}{\partial t} + \frac{\varepsilon}{2} \Delta u^\varepsilon + u^\varepsilon \nabla u^\varepsilon + g^\varepsilon = 0 \\ u^\varepsilon(T, x) = h(x) \\ x \in \mathbb{R}^d \end{cases} \quad (2.5)$$

If u^ε solves the problem above, $v^\varepsilon(t, x) = -u^\varepsilon(T - t, x)$ solves a Burgers Equation

$$\begin{cases} \frac{\partial v^\varepsilon}{\partial t} - \frac{\varepsilon}{2} \Delta v^\varepsilon + v^\varepsilon \nabla v^\varepsilon + g^\varepsilon = 0 \\ v^\varepsilon(0, x) = -h(x) \\ x \in \mathbb{R}^d \end{cases} \quad (2.6)$$

which is an important simplified model for turbulence and describes the motion of a compressible fluid with a viscosity parameter $\frac{\varepsilon}{2}$ under the influence of an external force g^ε .

The reference [21] is an example in the deterministic PDE literature where the behaviour of the above evolution problem when $\varepsilon \rightarrow 0$ is studied. It is known (cf [14]) that when G is an open bounded set of \mathbb{R}^3 with a smooth boundary ∂G , if $g \in L^\infty(0, T; L(G))$, there exists u^ε unique solution of a Burgers forward equation in the intersection of the functional spaces $L^\infty(0, T; L(G)) \cap L^2(0, T; W_0^{1,2}(G))$. When g is in $L^\infty(0, T; W_0^{3,2}(G))$, then u^ε will converge in $L^2(G)$ to the corresponding solution of the limiting equation on a small non-empty time interval as the viscosity parameter goes to zero [21].

One of our intentions is to study probabilistically (2.2), using the connection with the Forward Backward Stochastic Differential Equation (2.1), with the link provided via the identification

$$u^\varepsilon(t, x) = Y_t^{t, \varepsilon, x} \quad (2.7)$$

In [9] and [18], it is shown that a Large Deviation Principle holds for the couple $(X_s^{t, \varepsilon, x}, Y_s^{t, \varepsilon, x})_{t \leq s \leq T}$ in the case where the FBSDE (2.1) is decoupled, namely, when $f = f(s, x)$ and $\sigma = \sigma(s, x)$. But the fundamental a-posteriori property

$$Y_s^{t, \varepsilon, x}(\omega) = u^\varepsilon(s, X_s^{t, \varepsilon, x}(\omega)), s \in [t, T] \text{ a.s} \quad (2.8)$$

(see corollary 1.5 of [5]) turns the forward equation of (2.1) in an equation which only depends on $(X_s^{t,\varepsilon,x})_{t \leq s \leq T}$. With the work [1] on the so-called Freidlin-Wentzell Theory, which deals with diffusions given by

$$\begin{cases} dX_t^\varepsilon = b^\varepsilon(t, X_t^\varepsilon)dt + \sqrt{\varepsilon}\sigma^\varepsilon(t, X_t^\varepsilon)dB_t \\ X_0^\varepsilon = X_0 \end{cases} \quad (2.9)$$

with both the drift and the diffusion coefficient depending on ε , we are able to show that (2.1) obeys a Large Deviation Principle even in the coupled case, thus generalizing the results presented in [18]. We mention the work [10] where the authors do the asymptotic study of FBSDES with generalized Burgers type nonlinearities.

The main interest of this method is that we do not use PDE results to study the asymptotic behaviour of the Forward Backward Stochastic Differential System (2.1). In particular we have the possibility to study with probabilistic techniques the behaviour of a Burgers type equation when the viscosity parameter converges to zero. The other important point is the establishment of a Large Deviation Principle to the laws of the Forward and Backward Processes, even when the equations are coupled, unlike [9] and [18].

3 The Asymptotic Behaviour when $\varepsilon \rightarrow 0$

In order to establish our results, we make the following set of assumptions.

Assumption A

We say f, g, σ , and h satisfy (A.A) if there exists two constants $L, \Lambda > 0$ such that:

$$\begin{aligned} (A.1) \quad & \forall t \in [0, T], \forall (x, y, z), (\bar{x}, \bar{y}, \bar{z}) \in \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}: \\ & |f(t, x, y) - f(t, \bar{x}, \bar{y})| \leq L(|x - \bar{x}| + |y - \bar{y}|) \\ & |g(t, x, y, z) - g(t, \bar{x}, \bar{y}, \bar{z})| \leq L(|x - \bar{x}| + |y - \bar{y}| + |z - \bar{z}|) \\ & |h(x) - h(\bar{x})| \leq L(|x - \bar{x}|) \\ & |\sigma(t, x, y) - \sigma(t, \bar{x}, \bar{y})| \leq L(|x - \bar{x}| + |y - \bar{y}|) \end{aligned}$$

$$\begin{aligned} (A.2) \quad & \forall t \in [0, T], \forall (x, y, z), (\bar{x}, \bar{y}, \bar{z}) \in \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}: \\ & \langle x - \bar{x}, f(t, x, y) - f(t, \bar{x}, y) \rangle \leq L |x - \bar{x}|^2 \\ & \langle y - \bar{y}, f(t, x, y) - f(t, x, \bar{y}) \rangle \leq L |y - \bar{y}|^2 \end{aligned}$$

(A.3) $\forall t \in [0, T], \forall (x, y, z) \in \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$:

$$\begin{aligned} |f(t, x, y)| &\leq \Lambda(1 + |x| + |y|) \\ |g(t, x, y, z)| &\leq \Lambda(1 + |x| + |y| + |z|) \\ |\sigma(t, x, y)| &\leq \Lambda(1 + |x| + |y|) \\ |h(x)| &\leq \Lambda(1 + |x|) \end{aligned}$$

(A.4) $\forall t \in [0, T], \forall (x, y, z) \in \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$:

$u \mapsto f(t, u, y)$
 $v \mapsto g(t, x, v, z)$ are continuous mappings.

Under this set of hypothesis, *theorem 1.1* of [5] ensures that there exists a constant $C = C(L) > 0$, depending only on L , such that for every $T \leq C$ (1.1) admits a unique solution in \mathcal{M}_t .

Moreover, using *theorem 2.6* of [5], we have

$$\mathbb{P}(\forall s \in [t, T] : u^\varepsilon(s, X_s^{t, \xi}) = Y_s^{t, \xi}) = 1 \quad (3.1)$$

$$\mathbb{P} \otimes \mu(\{(\omega, s) \in \Omega \times [t, T] : |Z_s^{t, \varepsilon, x}(\omega)| \geq \Gamma_1\}) \quad (3.2)$$

where μ stands for the Lebesgue measure in the real line and Γ_1 is a constant which only depends on L, Λ, k, d, T .

By remark 2.7 of [5] we have that, for each $\varepsilon > 0$, u^ε only depends on the coefficients of the system (2.1), $f, g, \sqrt{\varepsilon}\sigma, h$. The fact that the dependence of $\Gamma_1 = \Gamma_1(L, \Lambda, k, d, T)$ determines that the properties (3.1)-(3.2) above hold uniformly in ε ; in particular, there exist continuous versions and uniformly bounded (in ε too) of $(Y_s^{t, \varepsilon, x}, Z_s^{t, \varepsilon, x})_{t \leq s \leq T}$.

We now assume more regularity on the coefficients of (1.1) in the next set of assumptions.

Assumption B

For $T \leq C$, we say that f, g, h, σ satisfy (A.B) if there exists three constants $\lambda, \Lambda, \gamma > 0$

(B.1) $\forall t \in [0, T], \forall (x, y, z) \in \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$:

$$\begin{aligned} |f(t, x, y)| &\leq \Lambda(1 + |y|) \\ |g(t, x, y, z)| &\leq \Lambda(1 + |y| + |z|) \\ |\sigma(t, x, y)| &\leq \Lambda \\ |h(x)| &\leq \Lambda \end{aligned}$$

(B.2) $\forall (t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$:
 $\langle \xi, a(t, x, y)\xi \rangle \geq \lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^d$ where $a(t, x, y) = \sigma \sigma^T(t, x, y)$.

(B.3) the function σ is continuous on its definition set.

(B.4) The function σ is differentiable with respect to x and y and its derivatives with respect to x and y are γ -Hölder in x and y , uniformly in t .

Under the set of assumptions (A.A) and (A.B), using *proposition 2.4* and *proposition B.6* of [5] and *theorem 2.9* of [6], one can prove that there exists two constants, κ, κ_1 , only depending on L, Λ, d, k, T (independent of ε) such that :

$$|u^\varepsilon(t, x)| \leq \kappa \quad (3.3)$$

$$u^\varepsilon \in C_b^{1,2}([0, T] \times \mathbb{R}^d) \quad (3.4)$$

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} |\nabla_x u^\varepsilon(t, x)| \leq \kappa_1 \quad (3.5)$$

and

$$Z_s^{t,\varepsilon,x} = \sqrt{\varepsilon} \nabla_x u^\varepsilon(t, X_s^{t,\varepsilon,x}) \sigma(s, X_s^{t,\varepsilon}, Y_s^{t,\varepsilon,x}) \quad (3.6)$$

where u^ε solves uniquely (2.2) in $C_b^{1,2}([0, T] \times \mathbb{R}^d)$, the space of C^1 functions with respect to the first variable and C^2 with respect to the second variable, with bounded derivatives.

All these facts can be proven probabilistically. The last claim and the properties (3.3), (3.4) and (3.6) are proved in [5]. Delarue delivers in the appendices of [5] probabilistic methods to obtain these regularity results, under assumptions (A.A) and (A.B) and over a small time enough duration, using *Malliavin* Calculus techniques. The estimate of the gradient (3.5) is established by a probabilistic scheme in [6], using a variant of the *Malliavin-Bismut* integration by parts formula proposed by Thalmaier [19] and applied in [20] to establish a gradient estimate of interior type for the solutions of a linear elliptic equation on a manifold.

Theorem 3.1. *Asymptotic Behaviour when $\varepsilon \rightarrow 0$.*

Under the previous assumptions (A), and (B), assuming further $T \leq C$, where $C = C(L, \gamma)$ depending on the Lipschitz constant L and on a constant γ that appears in the middle of the proof (see 3.13) we have

1. For each $s, t \in [0, T]$, $t \leq s$, the solution of (2.1), $(X_s^{t,\varepsilon,x}, Y_s^{t,\varepsilon,x}, Z_s^{t,\varepsilon,x})_{t \leq s \leq T}$ converges in \mathcal{M}_t , when $\varepsilon \rightarrow 0$ to $(X_s, Y_s, 0)_{t \leq s \leq T}$, where $(X_s, Y_s)_{t \leq s \leq T}$ solves the coupled system of differential equations:

$$\begin{cases} \dot{X}_s = f(s, X_s, Y_s) \\ \dot{Y}_s = -g(s, X_s, Y_s, 0), \quad t \leq s \leq T \\ X_t = x, Y_T = h(X_T) \end{cases} \quad (3.7)$$

2. Denoting $u(t, x) = Y_t^{t,x}$, the limit in $\varepsilon \rightarrow 0$ of $Y_t^{t,\varepsilon,x}$, the function u is a viscosity solution of

$$\begin{cases} \frac{\partial u^l}{\partial t} + \sum_{i=1}^d f_i(t, x, u(t, x)) \frac{\partial u^l}{\partial x_i}(t, x) + g^l(t, x, u(t, x), 0) = 0 \\ u^\varepsilon(t, x) = h(x); \quad x \in \mathbb{R}^d; \quad t \in [0, T]; \quad l = 1, \dots, k \end{cases} \quad (3.8)$$

3. The function u is bounded, continuous Lipschitz in x and uniformly continuous in time.

4. Furthermore, if $u \in C_b^{1,1}([0, T] \times \mathbb{R}^d)$, since that (3.7) has a unique continuous solution, due to the assumptions A and B, the function u is a classical solution of (3.8).

Proof. Given $\varepsilon, \varepsilon_1 > 0$, $x \in \mathbb{R}^d$, for $T \leq C$, if $(X_s^{t,\varepsilon,x}, Y_s^{t,\varepsilon,x}, Z_s^{t,\varepsilon,x})_{t \leq s \leq T}$ is the unique solution in \mathcal{M}_t of

$$\begin{cases} X_s^{t,\varepsilon,x} = x + \int_t^s f(r, X_r^{t,\varepsilon,x}, Y_r^{t,\varepsilon,x}) dr + \sqrt{\varepsilon} \int_t^s \sigma(r, X_r^{t,\varepsilon,x}, Y_r^{t,\varepsilon,x}) dB_r \\ Y_s^{t,\varepsilon,x} = h(X_T^{t,\varepsilon,x}) + \int_s^T g(r, X_r^{t,\varepsilon,x}, Y_r^{t,\varepsilon,x}, Z_r^{t,\varepsilon,x}) dr - \int_s^T Z_r^{t,\varepsilon,x} dB_r \\ x \in \mathbb{R}^d; \quad 0 \leq t \leq s \leq T \end{cases} \quad (3.9)$$

and $(X_s^{t,\varepsilon_1,x}, Y_s^{t,\varepsilon_1,x}, Z_s^{t,\varepsilon_1,x})_{t \leq s \leq T}$ is the unique solution in \mathcal{M}_t of

$$\begin{cases} X_s^{t,\varepsilon_1,x} = x + \int_t^s f(r, X_r^{t,\varepsilon_1,x}, Y_r^{t,\varepsilon_1,x}) dr + \sqrt{\varepsilon_1} \int_t^s \sigma(r, X_r^{t,\varepsilon_1,x}, Y_r^{t,\varepsilon_1,x}) dB_r \\ Y_s^{t,\varepsilon_1,x} = h(X_T^{t,\varepsilon_1,x}) + \int_s^T g(r, X_r^{t,\varepsilon_1,x}, Y_r^{t,\varepsilon_1,x}, Z_r^{t,\varepsilon_1,x}) dr - \int_s^T Z_r^{t,\varepsilon_1,x} dB_r \\ x \in \mathbb{R}^d; \quad 0 \leq t \leq s \leq T \end{cases} \quad (3.10)$$

Write, for sake of simplicity on the notations, with $\delta = \varepsilon, \varepsilon_1$

$$\begin{aligned} f^\delta(s) &= f(s, X_s^{t,\delta,x}, Y_s^{t,\delta,x}) \\ g^\delta(s) &= g(s, X_s^{t,\delta,x}, Y_s^{t,\delta,x}, Z_s^{t,\delta,x}) \\ \sigma^\delta(s) &= \sqrt{\delta}\sigma(s, X_s^{t,\delta,x}, Y_s^{t,\delta,x}) \\ h^\delta &= h(X_T^{t,\delta,x}) \end{aligned}$$

As in the proof of *theorem 1.2* of [5], using *Itô's Formula*:

$$\begin{aligned} &\mathbb{E} \sup_{t \leq s \leq T} |X_s^\varepsilon - X_s^{\varepsilon_1}|^2 \leq \mathbb{E} \int_t^T |\sigma^\varepsilon(s) - \sigma^{\varepsilon_1}(s)|^2 ds \\ &+ 2\mathbb{E} \sup_{t \leq s \leq T} \int_t^s \langle X_r^\varepsilon - X_r^{\varepsilon_1}, f^\varepsilon(r) - f^{\varepsilon_1}(r) \rangle dr \\ &+ 2\mathbb{E} \sup_{t \leq s \leq T} \int_t^s \langle X_r^\varepsilon - X_r^{\varepsilon_1}, (\sigma^\varepsilon(r) - \sigma^{\varepsilon_1}(r))dB_r \rangle \end{aligned} \quad (3.11)$$

Using *Burkholder-Davis-Gundy's inequalities*, there exists $\gamma > 0$ such that:

$$\begin{aligned} &\mathbb{E} \sup_{t \leq s \leq T} |X_s^\varepsilon - X_s^{\varepsilon_1}|^2 \leq \mathbb{E} \int_t^T |\sigma^\varepsilon(s) - \sigma^{\varepsilon_1}(s)|^2 ds \\ &+ 2\mathbb{E} \sup_{t \leq s \leq T} \int_t^s \langle X_r^\varepsilon - X_r^{\varepsilon_1}, f^\varepsilon(r) - f^{\varepsilon_1}(r) \rangle dr \\ &+ 2\gamma \mathbb{E} \left(\int_t^T |X_r^\varepsilon - X_r^{\varepsilon_1}|^2 |\sigma^\varepsilon(r) - \sigma^{\varepsilon_1}(r)|^2 dr \right)^{1/2} \end{aligned} \quad (3.12)$$

where $\gamma > 0$ only depends on the Lipschitz constant L and T .

Using the Lipschitz property (A.1), there exists $\gamma > 0$, eventually different, but only dependent on L, T such that:

$$\begin{aligned} &\mathbb{E} \sup_{t \leq s \leq T} |X_s^\varepsilon - X_s^{\varepsilon_1}|^2 \leq \gamma \left\{ \mathbb{E} \int_t^T (|X_s^\varepsilon - X_s^{\varepsilon_1}|^2 + |Y_s^\varepsilon - Y_s^{\varepsilon_1}|^2) ds \right. \\ &+ \mathbb{E} \int_t^T |\sigma^\varepsilon(s) - \sigma^{\varepsilon_1}(s)|^2 ds + 2|\sqrt{\varepsilon} - \sqrt{\varepsilon_1}|^2 \\ &\left. + 2\sqrt{T} \sup_{t \leq s \leq T} |X_s^\varepsilon - X_s^{\varepsilon_1}|^2 \right\} \end{aligned} \quad (3.13)$$

Then, assuming that $1 - 2\gamma\sqrt{T} \geq 0$

$$\begin{aligned} &(1 - 2\gamma\sqrt{T}) \sup_{t \leq s \leq T} |X_s^\varepsilon - X_s^{\varepsilon_1}|^2 \\ &\leq \gamma \left\{ \mathbb{E} \int_t^T (|X_s^\varepsilon - X_s^{\varepsilon_1}|^2 + |Y_s^\varepsilon - Y_s^{\varepsilon_1}|^2) ds + 2|\sqrt{\varepsilon} - \sqrt{\varepsilon_1}|^2 \right. \\ &\quad \left. + \mathbb{E} \int_t^T |\sigma^\varepsilon(s) - \sigma^{\varepsilon_1}(s)|^2 ds \right\} \end{aligned} \quad (3.14)$$

Using estimates as before, we get for some new $\gamma \geq 0$ eventually:

$$\begin{aligned} & \mathbb{E} \left(\sup_{t \leq s \leq T} |Y_s^\varepsilon - Y_s^{\varepsilon_1}|^2 \right) + \mathbb{E} \left(\int_t^T |Z_s^\varepsilon - Z_s^{\varepsilon_1}|^2 ds \right) \\ & \leq \gamma \left[\mathbb{E} |h^\varepsilon - h^{\varepsilon_1}|^2 + \mathbb{E} \left(\sup_{t \leq s \leq T} \int_t^T \langle Y_s^\varepsilon - Y_s^{\varepsilon_1}, g^\varepsilon(s) - g^{\varepsilon_1}(s) \rangle ds \right) \right] \end{aligned}$$

Using (3.13) and the assumption (A.1) modifying γ eventually:

$$\begin{aligned} & \mathbb{E} \sup_{t \leq s \leq T} |X_s^\varepsilon - X_s^{\varepsilon_1}|^2 + \mathbb{E} \sup_{t \leq s \leq T} |Y_s^\varepsilon - Y_s^{\varepsilon_1}|^2 + \mathbb{E} \int_t^T |Z_s^\varepsilon - Z_s^{\varepsilon_1}|^2 ds \\ & \leq \gamma \left\{ \mathbb{E} |h^\varepsilon - h^{\varepsilon_1}|^2 + \mathbb{E} \int_t^T |X_s^\varepsilon - X_s^{\varepsilon_1}|^2 ds + \mathbb{E} \int_t^T |Y_s^\varepsilon - Y_s^{\varepsilon_1}|^2 ds \right. \\ & \quad + \mathbb{E} \left(\int_t^T (|f^\varepsilon(s) - f^{\varepsilon_1}(s)| + |g^\varepsilon(s) - g^{\varepsilon_1}(s)|) ds \right)^2 \\ & \quad + \mathbb{E} \int_t^T |\sigma^\varepsilon(s) - \sigma^{\varepsilon_1}(s)|^2 ds + |\sqrt{\varepsilon} - \sqrt{\varepsilon_1}|^2 \\ & \quad \left. + \mathbb{E} \int_t^T |Z_s^\varepsilon - Z_s^{\varepsilon_1}| (|Y_s^\varepsilon - Y_s^{\varepsilon_1}| + |X_s^\varepsilon - X_s^{\varepsilon_1}|) ds \right\} \end{aligned}$$

So, there exist $C^* \leq C$ and Γ , only depending on L and γ_1 , such that, for $T - t \leq C^*$

$$\begin{aligned} & \mathbb{E} \sup_{t \leq s \leq T} |X_s^\varepsilon - X_s^{\varepsilon_1}|^2 + \mathbb{E} \sup_{t \leq s \leq T} |Y_s^\varepsilon - Y_s^{\varepsilon_1}|^2 + \mathbb{E} \int_t^T |Z_s^\varepsilon - Z_s^{\varepsilon_1}|^2 ds \\ & \leq \gamma \left\{ \mathbb{E} |h^\varepsilon - h^{\varepsilon_1}|^2 + \mathbb{E} \int_t^T |\sigma^\varepsilon(s) - \sigma^{\varepsilon_1}(s)|^2 ds + |\sqrt{\varepsilon} - \sqrt{\varepsilon_1}|^2 \right. \\ & \quad \left. + \mathbb{E} \left(\int_t^T |Y_s^\varepsilon - Y_s^{\varepsilon_1}| + |g^\varepsilon(s) - g^{\varepsilon_1}(s)| ds \right)^2 \right\} \end{aligned} \tag{3.15}$$

We can repeat the argument in $[T - 2C^*; T - C^*]$ and recurrently we get this important a-priori estimate (3.15) valid for all $[t, T]$, with some (possibly different) constant $\gamma > 0$ only depending on L .

Now, from (A.1), the following estimate holds:

$$\mathbb{E} |h^\varepsilon - h^{\varepsilon_1}|^2 \leq L^2 \mathbb{E} \sup_{t \leq s \leq T} |X_s^\varepsilon - X_s^{\varepsilon_1}|^2 \tag{3.16}$$

One can also derive the estimate

$$\mathbb{E} \int_t^T \left(|\sqrt{\varepsilon} \sigma^\varepsilon(s) - \sqrt{\varepsilon_1} \sigma^{\varepsilon_1}(s)|^2 \right) ds \leq c_1 |\sqrt{\varepsilon} - \sqrt{\varepsilon_1}|^2$$

$$+c_2 \mathbb{E} \int_t^T (|X_s^\varepsilon - X_s^{\varepsilon_1}|^2 + |Y_s^\varepsilon - Y_s^{\varepsilon_1}|^2) ds \quad (3.17)$$

where $c_1, c_2 > 0$ only depending on T, L and on the bound of $|\sigma|$.

Furthermore, using *Jensen's inequality* and the Lipschitz property of g , for some c_3 modified eventually along the various steps,

$$\begin{aligned} & \mathbb{E} \left(\int_t^T |Y_s^\varepsilon - Y_s^{\varepsilon_1}| + |g^\varepsilon(s) - g^{\varepsilon_1}(s)| ds \right)^2 \\ & \leq \mathbb{E} \int_t^T |Y_s^\varepsilon - Y_s^{\varepsilon_1}|^2 + |g^\varepsilon - g^{\varepsilon_1}|^2 + 2|Y_s^\varepsilon - Y_s^{\varepsilon_1}| |g^\varepsilon - g^{\varepsilon_1}| ds \\ & \leq c_3 \mathbb{E} \left[\int_t^T |X_s^\varepsilon - X_s^{\varepsilon_1}|^2 + |Y_s^\varepsilon - Y_s^{\varepsilon_1}|^2 + |Z_s^\varepsilon - Z_s^{\varepsilon_1}|^2 \right] \\ & \leq c_3 \left\{ \mathbb{E} \sup_{t \leq s \leq T} |X_s^\varepsilon - X_s^{\varepsilon_1}|^2 + \mathbb{E} \sup_{t \leq s \leq T} |Y_s^\varepsilon - Y_s^{\varepsilon_1}|^2 + \mathbb{E} \sup_{t \leq s \leq T} |Z_s^\varepsilon - Z_s^{\varepsilon_1}|^2 \right\} \end{aligned} \quad (3.18)$$

Using (3.6) and the boundedness of $|\sigma|$, we have

$$\begin{aligned} & \mathbb{E} \int_t^T |Z_s^\varepsilon|^2 ds \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \text{ and} \\ & \lim_{|\varepsilon - \varepsilon_1| \rightarrow 0} \mathbb{E} \int_t^T |Z_s^\varepsilon - Z_s^{\varepsilon_1}|^2 ds = 0 \end{aligned} \quad (3.19)$$

Furthermore, by *Burkholder-Davis-Gundy's inequalities*, there exists a constant $\gamma > 0$, eventually new, such that:

$$\begin{aligned} & \mathbb{E} \sup_{t \leq s \leq T} |X_s^\varepsilon - X_s^{\varepsilon_1}|^2 + \mathbb{E} \sup_{t \leq s \leq T} |Y_s^\varepsilon - Y_s^{\varepsilon_1}|^2 + \mathbb{E} \int_t^T |Z_s^\varepsilon - Z_s^{\varepsilon_1}|^2 ds \\ & \leq \gamma \mathbb{E} \int_t^T |X_s^\varepsilon - X_s^{\varepsilon_1}|^2 + |Y_s^\varepsilon - Y_s^{\varepsilon_1}|^2 + |Z_s^\varepsilon - Z_s^{\varepsilon_1}|^2 ds \\ & \leq \gamma \mathbb{E} \int_t^T \sup_{t \leq r \leq s} |X_r^\varepsilon - X_r^{\varepsilon_1}|^2 + \sup_{t \leq r \leq s} |Y_r^\varepsilon - Y_r^{\varepsilon_1}|^2 + \sup_{t \leq r \leq s} |Z_r^\varepsilon - Z_r^{\varepsilon_1}|^2 ds \end{aligned} \quad (3.20)$$

Moreover, for some $\gamma_1, \gamma_2 > 0$, using (3.18)- (3.21)

$$\begin{aligned} & \mathbb{E} \sup_{t \leq s \leq T} |X_s^\varepsilon - X_s^{\varepsilon_1}|^2 + \mathbb{E} \sup_{t \leq s \leq T} |Y_s^\varepsilon - Y_s^{\varepsilon_1}|^2 + \mathbb{E} \int_t^T |Z_s^\varepsilon - z_s^{\varepsilon_1}|^2 ds \\ & \leq \gamma_1 |\sqrt{\varepsilon} - \sqrt{\varepsilon_1}|^2 + \gamma_2 \mathbb{E} \int_t^T \sup_{t \leq r \leq s} |X_r^\varepsilon - X_r^{\varepsilon_1}|^2 + \sup_{t \leq r \leq s} |Y_r^\varepsilon - Y_r^{\varepsilon_1}|^2 \end{aligned}$$

$$+ \sup_{t \leq r \leq s} |Z_r^\varepsilon - Z_r^{\varepsilon_1}|^2 ds \quad (3.21)$$

Using *Gronwall's inequality*,

$$\begin{aligned} \mathbb{E} \int_t^T \sup_{t \leq r \leq s} |X_r^\varepsilon - X_r^{\varepsilon_1}|^2 + \sup_{t \leq r \leq s} |Y_s^\varepsilon - Y_s^{\varepsilon_1}|^2 &\leq \\ &\leq C_1 |\sqrt{\varepsilon} - \sqrt{\varepsilon_1}|^2 \mathbb{E} \sup_{t \leq s \leq T} |Z_s^\varepsilon - Z_s^{\varepsilon_1}|^2 \rightarrow 0 \text{ as } |\varepsilon - \varepsilon_1| \rightarrow 0. \end{aligned} \quad (3.22)$$

for some $C_1 > 0$ only depending on L (independent of ε), once

$\mathbb{E} \sup_{t \leq s \leq T} |Z_s^\varepsilon - Z_s^{\varepsilon_1}|^2$ is bounded, by the results (3.5) and (3.6) .

We conclude, by the previous estimates, that the pair $(X_s^{t,\varepsilon}, Y_s^{t,\varepsilon})_{t \leq s \leq T}$ form a Cauchy sequence and therefore converges in $S_t^2(\mathbb{R}^d) \times S_t^2(\mathbb{R}^k)$. Denote $(X_s, Y_s)_{t \leq s \leq T}$ its limit. $(X_s, Y_s, 0)_{t \leq s \leq T}$ is the limit in \mathcal{M}_t of $(X_s^{t,\varepsilon}, Y_s^{t,\varepsilon}, Z_s^{t,\varepsilon,x})_{t \leq s \leq T}$ when $\varepsilon \rightarrow 0$.

If we consider the forward equation in (2.1) and if we take the limit pointwise when $\varepsilon \rightarrow 0$, we have

$$X_s^t = x + \int_t^s f(r, X_r, Y_r) dr \text{ a.s} \quad (3.23)$$

where we have used the boundedness of σ and the continuity of f .

Similarly, we can take the limit on the backward equation when $\varepsilon \rightarrow 0$.

Using the continuity of the functions h, g and

$$\mathbb{E} \left(\int_s^T Z_r^{t,\varepsilon} dB_r \right)^2 = \mathbb{E} \int_s^T |Z_r^{t,\varepsilon}|^2 dr \rightarrow 0 \text{ as } \varepsilon \rightarrow 0$$

which implies $\int_s^T Z_r^{t,\varepsilon} dB_r \rightarrow 0$, a.s \mathbb{P} , we have

$$Y_s^t = h(X_T) + \int_s^T g(r, X_r, Y_r, 0) dr \text{ a.s} \quad (3.24)$$

In conclusion, $(X_s, Y_s)_{t \leq s \leq T}$ solves the following deterministic problem of ordinary (coupled) differential equations, almost surely,

$$\begin{cases} \dot{X}_s = f(s, X_s, Y_s) \\ \dot{Y}_s = -g(s, X_s, Y_s, 0), \quad t \leq s \leq T \\ X_t = x, Y_T = h(X_T) \end{cases} \quad (3.25)$$

Given $t, t' \in [0, T]$, $x, x' \in \mathbb{R}^d$, consider $(X_s^{t,\varepsilon,x}, Y_s^{t,\varepsilon,x}, Z_s^{t,\varepsilon,x})_{t \leq s \leq T}$ the unique solution in \mathcal{M}_t

$$\begin{cases} X_s^{t,\varepsilon,x} = x + \int_t^s f(r, X_r^{t,\varepsilon,x}, Y_r^{t,\varepsilon,x})dr + \sqrt{\varepsilon} \int_t^s \sigma(r, X_r^{t,\varepsilon,x}, Y_r^{t,\varepsilon,x})dB_r \\ Y_s^{t,\varepsilon,x} = h(X_s^{t,\varepsilon,x}) + \int_s^T g(r, X_r^{t,\varepsilon,x}, Y_r^{t,\varepsilon,x}, Z_r^{t,\varepsilon,x})dr - \int_s^T Z_r^{t,\varepsilon,x}dB_r \\ t \leq s \leq T \end{cases} \quad (3.26)$$

extended to the whole interval $[0, T]$, putting

$$\forall 0 \leq s \leq t \quad X_s^{t,\varepsilon,x} = x; \quad Y_s^{t,\varepsilon,x} = Y_t^{t,\varepsilon,x}; \quad Z_s^{t,\varepsilon,x} = 0 \quad (3.27)$$

and consider $(X_s^{t',\varepsilon,x'}, Y_s^{t',\varepsilon,x'}, Z_s^{t',\varepsilon,x'})_{t' \leq s \leq T}$ the unique solution in $\mathcal{M}_{t'}$

$$\begin{cases} X_s^{t',\varepsilon,x'} = x' + \int_{t'}^s f(r, X_r^{t',\varepsilon,x'}, Y_r^{t',\varepsilon,x'})dr + \sqrt{\varepsilon} \int_{t'}^s \sigma(r, X_r^{t',\varepsilon,x'}, Y_r^{t',\varepsilon,x'})dB_r \\ Y_s^{t',\varepsilon,x'} = h(X_s^{t',\varepsilon,x'}) + \int_s^T g(r, X_r^{t',\varepsilon,x'}, Y_r^{t',\varepsilon,x'}, Z_r^{t',\varepsilon,x'})dr - \int_s^T Z_r^{t',\varepsilon,x'}dB_r \\ t' \leq s \leq T \end{cases} \quad (3.28)$$

extended to the whole interval $[0, T]$, putting

$$\forall 0 \leq s \leq t' \quad X_s^{t',\varepsilon,x'} = x'; \quad Y_s^{t',\varepsilon,x'} = Y_{t'}^{t',\varepsilon,x'}; \quad Z_s^{t',\varepsilon,x'} = 0 \quad (3.29)$$

Using the estimate (3.15), such as in the *corollary 1.4* of [5], we are lead to

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq s \leq T} |X_s^{t,\varepsilon,x} - X_s^{t',\varepsilon,x'}|^2 + \mathbb{E} \sup_{0 \leq s \leq T} |Y_s^{t,\varepsilon,x} - Y_s^{t',\varepsilon,x'}|^2 \\ & + \mathbb{E} \int_0^T |Z_s^{t,\varepsilon,x} - Z_s^{t',\varepsilon,x'}|^2 ds \leq \alpha |x - x'|^2 + \beta(1 + |x|^2) |t - t'|^2 \end{aligned} \quad (3.30)$$

where $\alpha, \beta > 0$ are constants only depending on L, Λ .

Finally,

$$|u^\varepsilon(t, x) - u^\varepsilon(t', x')|^2 \leq \alpha |x - x'|^2 + \beta(1 + |x|^2) |t - t'|^2 \quad (3.31)$$

which proves that $(u^\varepsilon)_{\varepsilon>0}$ is a family of equicontinuous maps on every compact set of $[0, T] \times \mathbb{R}^d$. We can apply *Arzela's Ascoli Theorem* and conclude that the convergence of u^ε to u , where $u(t, x) = Y_t^{t,x}$, is uniform in $[0, T] \times K$, for every K compact subset of \mathbb{R}^d .

Taking the limit in $\varepsilon \rightarrow 0$ in (3.31) we get

$$|u(t', x') - u(t, x)|^2 \leq \alpha |x - x'|^2 + \beta(1 + |x|^2) |t - t'|^2 \quad (3.32)$$

for all $(t, x), (t', x') \in [0, T] \times \mathbb{R}^d$, which proves that the function u , limit in $\varepsilon \rightarrow 0$, is Lipschitz continuous in x and uniformly continuous in t . The boundedness of u is given by (3.3), since this bound is uniform in ε .

Using *theorem 5.1* of [15] we deduce clearly that u^ε is a viscosity solution in $[0, T] \times \mathbb{R}^d$ of (2.2).

Since the coefficients of the quasilinear parabolic system are Lipschitz continuous, by the property of the compact uniform convergence of viscosity solutions for quasilinear parabolic equations (see [4] for details), we conclude that u is a viscosity solution of (3.8).

Moreover, let $v : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^k$ be a $C_b^{1,1}([0, T], \mathbb{R}^k)$ solution, continuous Lipschitz in x and uniformly continuous in t for (3.8). Fixing $(t, x) \in [0, T] \times \mathbb{R}^d$, we take the following function:

$$\begin{aligned} \psi : [t, T] &\rightarrow \mathbb{R}^k \\ \psi(s) &:= v(s, X_s^{t,x}). \end{aligned}$$

Computing its time derivative:

$$\begin{aligned} \frac{d\psi}{ds}(s) &= \frac{\partial v}{\partial s}(s, X_s^{t,x}) + \sum_{i=1}^d \frac{\partial v}{\partial x_i}(s, X_s^{t,x}) \frac{\partial (X_s^{t,x})}{\partial t} \\ &= \frac{\partial v}{\partial s}(s, X_s^{t,x}) + \sum_{i=1}^d \frac{\partial v}{\partial x_i}(s, X_s^{t,x}) f(s, X_s^{t,x}, Y_s^{t,x}) \\ &= -g(s, X_s^{t,x}, v(s, X_s^{t,x}), 0) \end{aligned}$$

$$\psi(T) = v(T, X_T^{t,x}) = h(x)$$

As a consequence, $v(t, x) = v(t, X_t^{t,x}) = u(t, x)$, under the hypothesis of uniqueness of solution for the system of ordinary differential equations (3.7).

So, under these hypothesis, we have a uniqueness property of solution for (3.8) in the class of $C_b^{1,1}([0, T] \times \mathbb{R}^d)$, which are Lipschitz continuous in x and uniformly continuous in t . □

4 A Large Deviations Principle

In this section we state a Large Deviation Principle for the laws of the processes $(X_s^{t,\varepsilon,x}, Y_s^{t,\varepsilon,x})_{t \leq s \leq T}$ when the parameter $\varepsilon \rightarrow 0$.

The main property employed to establish the Large Deviation Principle for $(X_s^{t,\varepsilon,x}, Y_s^{t,\varepsilon,x})_{t \leq s \leq T}$ is

$$Y_s^{t,\varepsilon,x} = u^\varepsilon(s, X_s^{t,\varepsilon,x}) \text{ a.s.} \quad (4.1)$$

That property is the key to generalize the Large Deviation Principle proved in [18] where the FBSDE is decoupled, namely when $f(t, x, y) = f(t, x)$ and $\sigma(t, x, y) = \sigma(t, x)$.

For our purposes, we need the following results of Large Deviations Theory.

Theorem 4.1. Freidlin-Wentzell Estimates ([1],[2])

Given $t \in [0, T]$ and $x \in \mathbb{R}^d$, consider $b : [t, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma : [t, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ Lipschitz continuous functions, with sublinear growth and bounded in time. For each $\varepsilon > 0$, let $b^\varepsilon : [t, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma^\varepsilon : [t, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ be Lipschitz continuous and with sublinear growth, such that:

$$\lim_{\varepsilon \rightarrow 0} |b^\varepsilon - b| = \lim_{\varepsilon \rightarrow 0} |\sigma^\varepsilon - \sigma| = 0 \quad (4.2)$$

uniformly in the compact sets of $[t, T] \times \mathbb{R}^d$.

The process $(X_s^\varepsilon)_{t \leq s \leq T}$, the unique strong solution of the stochastic differential equation:

$$\begin{cases} dX_s^\varepsilon = b^\varepsilon(s, X_s^\varepsilon)ds + \sqrt{\varepsilon}\sigma^\varepsilon(s, X_s^\varepsilon)dB_s \\ X_t^\varepsilon = x \end{cases} \quad (4.3)$$

satisfies a Large Deviations Principle in $C_x([t, T], \mathbb{R}^d)$, the space of the continuous functions $\phi : [t, T] \rightarrow \mathbb{R}^d$ having origin in x with the good rate function:

$$I(g) = \inf \left\{ \frac{1}{2} \|\varphi\|_{H^1}^2 : g \in H^1([t, T], \mathbb{R}^d) \text{ } g_s = x + \int_t^s b(r, g_r)dr + \int_t^s \sigma(r, g_r)\dot{\varphi}_r dr \right\} \quad (4.4)$$

This means that the level sets of I are compact and

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in F) \leq - \inf_{\psi \in F} I(\psi) \quad (4.5)$$

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in G) \geq - \inf_{\psi \in G} I(\psi) \quad (4.6)$$

for every closed set $F \in C_x([t, T], \mathbb{R}^d)$ and for every open set $G \in C_x([t, T], \mathbb{R}^d)$

The next result is a very important tool in Large Deviations Theory, to transfer Large Deviations Principles from one topological space to another. We present a simple version of the Contraction Principle.

Theorem 4.2. Contraction Principle ([7])

If $f : X \rightarrow Y$ is a continuous mapping from a topological vector space X to a metric space (Y, d) and $\{\mu_\varepsilon\}_{\varepsilon > 0}$ satisfies a Large Deviations Principle with a good rate function $I : X \rightarrow [0, \infty]$, and for each $\varepsilon > 0$ if $f_\varepsilon : X \rightarrow Y$ is a family of continuous functions, uniformly convergent to f in all compact sets of X , we have that

$\{\mu_\varepsilon \circ f_\varepsilon^{-1}\}_{\varepsilon > 0}$ satisfies a Large Deviation Principle with the good rate function:

$$J(y) = \inf \{I(x) : x \in X \text{ and } f(x) = y\}$$

Remark: We define the infimum over the empty set to be $+\infty$.

Under conditions (A.A) and (A.B) are in force, for $T \leq C$, we have

Theorem 4.3. A Large Deviations Principle

When $\varepsilon \rightarrow 0$, $(X_s^{t, \varepsilon})_{t \leq s \leq T}$ obey a Large Deviation Principle in $C([t, T], \mathbb{R}^d)$ with the good rate function

$$I(g) = \inf \left\{ \frac{1}{2} \int_t^T |\dot{\varphi}_s|^2 ds : g \in H^1([t, T], \mathbb{R}^d), \right.$$

$$\left. g_s = x + \int_t^s f(r, g_r, u(r, g_r)) dr + \int_t^s \sigma(r, g_r, u(r, g_r)) \dot{\varphi}_r dr \text{ } s \in [t, T] \right\} \quad (4.7)$$

for $\varphi \in C([t, T], \mathbb{R}^d)$

and $(Y_s^{t, \varepsilon})_{t \leq s \leq T}$ obey a LDP in $C([t, T], \mathbb{R}^k)$ with the good rate function

$$J(\psi) = \inf \left\{ I(\varphi) : F(\varphi) = \psi \text{ if } \psi \in H^1([t, T], \mathbb{R}^d) \right\} \quad (4.8)$$

where $F(\varphi)(s) = u(s, \varphi_s)$ for all $\varphi \in C([t, T], \mathbb{R}^d)$

Proof. Since $Y_s^{t,\varepsilon,x} = u^\varepsilon(s, X_s^{t,\varepsilon,x})$, the first equation on the FBSDE (2.1) is in the differential form given by (we omit the indices)

$$\begin{cases} dX_s^\varepsilon = b^\varepsilon(s, X_s^\varepsilon)ds + \sqrt{\varepsilon}\sigma_1^\varepsilon(s, X_s^\varepsilon)dB_s; & t \leq s \leq T \\ X_t^\varepsilon = x \end{cases} \quad (4.9)$$

if we write $b^\varepsilon(s, X_s^\varepsilon) = f(s, X_s^\varepsilon, u^\varepsilon(s, X_s^\varepsilon))$ and $\sigma_1^\varepsilon(s, X_s^\varepsilon) = \sigma(s, X_s^\varepsilon, u^\varepsilon(s, X_s^\varepsilon))$.

We have therefore a typical setting in which we can apply *theorem 4.1*.

Define:

$$\begin{aligned} b^\varepsilon &: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d \\ b^\varepsilon(t, x) &= f(t, x, u^\varepsilon(t, x)) \\ \sigma_1^\varepsilon &: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d} \\ \sigma_1^\varepsilon(t, x) &= \sigma(t, x, u^\varepsilon(t, x)) \end{aligned}$$

We notice that b^ε and σ_1^ε are clearly Lipschitz continuous, with sub-linear growth.

Since $\lim_{\varepsilon \rightarrow 0} |u^\varepsilon(t, x) - u(t, x)| = 0$ uniformly in all compact sets of $[0, T] \times \mathbb{R}^d$ as we remarked in the *theorem 3.1* before, by the Lipschitz property of the coefficients of the FBSDE (2.1)- assumption (A.1)

$$\lim_{\varepsilon \rightarrow 0} |b^\varepsilon(t, x) - b(t, x)| = \lim_{\varepsilon \rightarrow 0} |\sigma^\varepsilon(t, x) - \sigma(t, x)| = 0. \quad (4.10)$$

uniformly in all compact sets of $[0, T] \times \mathbb{R}^d$,

where b and σ are defined by the obvious way:

$$b(t, x) = f(t, x, u(t, x)) \text{ and } \sigma(t, x) = \sigma(t, x, u(t, x)).$$

By the standard existence and uniqueness theory for ordinary differential equations, since b and σ_1 are Lipschitz continuous, the problem

$$\begin{cases} \dot{g}_s = b(s, g_s)ds + \sigma(s, g_s)\dot{h}_s; & h \in H^1([t, T], \mathbb{R}^d) \\ g_t = x \end{cases} \quad (4.11)$$

has a unique solution in $C([t, T], \mathbb{R}^d)$; we will denote it $g = S_x(h)$.

We have defined an operator $S_x : H^1([t, T], \mathbb{R}^d) \rightarrow C([t, T], \mathbb{R}^d)$ which is uniformly continuous (i.e. continuous for the supremum norm). We can therefore apply *theorem 4.1* and state that $(\mathbb{P} \circ (X^\varepsilon)^{-1})_{\varepsilon > 0}$ obey a Large Deviation Principle with the good rate function

$$I(g) = \inf \left\{ \frac{1}{2} \int_t^T |\dot{\varphi}_s|^2 ds : g \in H^1([t, T], \mathbb{R}^d), \right.$$

$$g_s = x + \int_t^s f(r, g_r, u(r, g_r))dr + \int_t^s \sigma(r, g_r, u(r, g_r))\dot{\varphi}_r dr \quad s \in [t, T] \Big\} \quad (4.12)$$

for $\varphi \in C([t, T], \mathbb{R}^d)$

In what follows, in order to prove a LDP to $(Y_s^\varepsilon)_{t \leq s \leq T}$ we consider the following operator:

$$\begin{aligned} F^\varepsilon : C([t, T], \mathbb{R}^d) &\rightarrow C([t, T], \mathbb{R}^k) \\ F^\varepsilon(\varphi)(s) &= u^\varepsilon(s, \varphi_s) \end{aligned} \quad (4.13)$$

We observe that $Y_s^\varepsilon = F^\varepsilon(X_s^\varepsilon)$ for all $s \in [t, T]$.

To establish a Large Deviation Principle for $(\mathbb{P} \circ (Y^\varepsilon)^{-1})_{\varepsilon > 0}$ we will proceed using the Contraction Principle in the form presented in *theorem 4.2*.

Proving *the continuity of F^ε* :

Let $\varepsilon > 0$ and $x \in C([t, T], \mathbb{R}^d)$. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $C([t, T], \mathbb{R}^d)$ converging to x in the uniform norm. Fix $\delta > 0$. Since $\|x_n - x\|_\infty \rightarrow 0$, there exists $M > 0$ such that $\|x\|_\infty, \|x_n\|_\infty \leq M$.

Due to *theorem 4.1*, we know that u^ε is a continuous function in $[0, T] \times \mathbb{R}^d$ and u^ε is uniformly continuous in $[t, T] \times K$ where $K = \overline{B(0, M)} \subset \mathbb{R}^d$.

There exists $\eta > 0$ such that for $s, s_1 \in [t, T]$ and $z, z_1 \in K$ $|s - s_1| < \eta$ and $|z - z_1| < \eta$, we have $|u^\varepsilon(s, z) - u^\varepsilon(s_1, z_1)| < \delta$.

Since $x_n \rightarrow x$ in $C([t, T], \mathbb{R}^d)$, fix $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\|x_n - x\|_\infty < \eta$.

For all $r \in [t, T]$ and for all $n \geq n_0$ $x_n(r), x(r) \in K$ and

$$|u^\varepsilon(r, x(r)) - u^\varepsilon(r, x_n(r))| < \delta.$$

So, we conclude that $F^\varepsilon(x_n) \rightarrow F^\varepsilon(x)$, which proves the continuity of F^ε in the point $x \in C([t, T], \mathbb{R}^d)$.

The next step consists in showing the uniform convergence, in the compact sets of $C([t, T], \mathbb{R}^d)$ of F^ε to F , when $\varepsilon \rightarrow 0$, where $F(\varphi)(s) = u(s, \varphi_s)$ for all $\varphi \in C([t, T], \mathbb{R}^d)$.

Due to the point 1 of the *theorem 3.1*, we know that for all $x \in \mathbb{R}^d$

$$\mathbb{E} \sup_{t \leq s \leq T} |Y_s^{\varepsilon, t, x} - Y_s^{t, x}|^2 + \mathbb{E} \int_t^T |Z_s^{\varepsilon, t, x}|^2 ds \rightarrow 0. \quad (4.14)$$

Consider K a compact set of $C([t, T], \mathbb{R}^d)$ and let

$$A := \{\varphi_s : \varphi \in K, s \in [t, T]\}.$$

It is clear that A is a compact set of \mathbb{R}^d . Using (4.14) we observe that

$$\begin{aligned}
\sup_{\varphi \in K} \|F^\varepsilon(\varphi) - F(\varphi)\|_\infty^2 &= \sup_{\varphi \in K} \sup_{s \in [t, T]} |u^\varepsilon(s, \varphi_s) - u(s, \varphi_s)|^2 \\
&= \sup_{\varphi \in K} \sup_{s \in [t, T]} |Y_s^{\varepsilon, s, \varphi_s} - Y_s^{s, \varphi_s}|^2 \\
&\leq \sup_{x \in A} \sup_{s \in [t, T]} |Y_s^{\varepsilon, s, x} - Y_s^{s, x}|^2 \rightarrow 0 \text{ as } \varepsilon \rightarrow 0
\end{aligned} \tag{4.15}$$

using the uniform convergence of u^ε to u in the compact sets of $[0, T] \times \mathbb{R}^d$.

Using *theorem 4.2*, we conclude that $(\mathbb{P} \circ (Y^\varepsilon)^{-1})_{\varepsilon > 0}$ satisfies a LDP principle with the good rate function

$$J(\psi) = \inf \left\{ I(\varphi) : F(\varphi) = \psi; \varphi \in H^1([t, T], \mathbb{R}^d) \right\} \tag{4.16}$$

for $\psi \in C([t, T], \mathbb{R}^k)$

□

5 Remarks and conclusions

The results presented in this work - *theorem 3.1* and *theorem 4.3*, were stated under the assumptions (A.A) and (A.B), which ensure existence and uniqueness of solution of (2.1) for a local time $T \leq C$, where C is a certain constant only depending on the Lipschitz constant L . Under these assumption, our results and the existence and uniqueness of solution for the FBSDE (2.1) ([5], [14]) do not depend on results for PDE's but only on probabilistic arguments. However, it is possible to extend *theorem 3.1* and *theorem 4.3* to global time. In order to do it and to prove the important properties (3.3) - (3.5) in [5], results of deterministic quasilinear parabolic partial differential equations [11] are used. If we maintain the assumption of smoothness on the coefficients of the FBSDE (2.1) (B.4), the *Four Step Scheme Methodology* ensures existence and uniqueness of solution for (2.2), and if we remove this requirement of smoothness on the coefficients of the system (2.1), a regularization argument in [5] is used in order to prove it. In this setting, we can also conclude the claims (2) and (3) of *theorem 3.1* and *theorem 4.3*.

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